

# Lévy Processes on $U_q(\mathfrak{g})$ as Infinitely Divisible Representations<sup>\*†‡</sup>

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## Abstract

Lévy processes on bialgebras are families of infinitely divisible representations. We classify the generators of Lévy processes on the compact forms of the quantum algebras  $U_q(\mathfrak{g})$ , where  $\mathfrak{g}$  is a simple Lie algebra. Then we show how the processes themselves can be reconstructed from their generators and study several classical stochastic processes that can be associated to these processes.

## 1 Introduction

Lévy processes on involutive bialgebras made their first appearance in a model of the laser studied by von Waldenfels, cf. [W84]. Their algebraic framework was formulated in [ASW88], for their general theory see [Sch93].

Let  $\mathcal{B}$  be an involutive bialgebra, i.e. an involutive unital associative algebra over  $\mathbb{C}$  with two  $*$ -homomorphisms  $\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$  and  $\varepsilon : \mathcal{B} \rightarrow \mathbb{C}$  satisfying

$$\begin{aligned} (\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta \\ (\varepsilon \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}, \end{aligned}$$

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and let  $\pi_i = (\pi_i, H_i, \Omega_i)$  be (cyclic) representations of  $\mathcal{B}$  on some pre-Hilbert spaces  $H_i$  with the vacuum vectors  $\Omega_i \in H_i$ ,  $i = 1, \dots, n$ . Then the product of these representations is the representation

$$\Pi_{i=1}^n \pi_i = ((\pi_1 \otimes \dots \otimes \pi_n) \circ \Delta^{(n)}, V_1 \otimes \dots \otimes V_n, \Omega_1 \otimes \dots \otimes \Omega_n),$$

where  $\Delta^{(n)}$  is defined by  $\Delta^{(1)} = \text{id}$ ,  $\Delta^{(2)} = \Delta$  and  $\Delta^{(n)} = (\Delta \otimes \text{id}^{\otimes(n-2)}) \circ \Delta^{(n-1)}$  for  $n \geq 3$ . A representation  $\pi$  is called infinitely divisible, if for any integer  $n \geq 1$  there exists a representation  $\pi^{(n)}$  such that

$$\pi \cong \Pi_{i=1}^n \pi_i, \quad \text{with} \quad \pi_i = \pi^{(n)} \quad \text{for } i = 1, \dots, n.$$

Two representations  $\pi = (\pi, H, \Omega)$  and  $\pi' = (\pi', H', \Omega')$  are considered as equivalent, if the vacuum expectations coincide, i.e.

$$\pi \cong \pi' \quad \Leftrightarrow \quad \langle \Omega, \pi(a) \Omega \rangle = \langle \Omega', \pi'(a) \Omega' \rangle \quad \text{for all } a \in \mathcal{B}.$$

All Lévy processes on  $*$ -bialgebras (see Definition 2.1) define examples of infinitely divisible representations of these  $*$ -bialgebras, see [Sch90] and also below.

Lévy processes on  $*$ -bialgebras have also been investigated in relation with probability theory. A class of stochastic processes with rather surprising properties, the so-called Azéma martingales, have been shown to be classical versions (see Section 5) of Lévy processes on a non-commutative, non-cocommutative involutive bialgebra, see [Sch93] and the references therein.

In this paper we will be interested in Lévy processes on the compact forms  $\mathcal{U}$  of the Drinfeld-Jimbo quantum enveloping algebras  $U_q(\mathfrak{g})$  corresponding to the simple Lie algebras  $\mathfrak{g}$  in the formulation of Jimbo, cf. [J85]. In Section 2, we recall the definition of these  $*$ -bialgebras as well as the definition of Lévy processes and some of their elementary properties. Section 3 contains the first main result of this paper, the characterisation of the Lévy processes on  $\mathcal{U}$  in terms of their generators. It turns out that all generators are of the form

$$\psi(u) = \langle \Omega, (\rho(u) - \varepsilon(u)) \Omega \rangle, \quad \text{for all } u \in \mathcal{U},$$

where  $\rho$  is a unitary representation of  $\mathcal{U}$  on some pre-Hilbert space  $D$ ,  $\Omega$  some vector in  $D$  and  $\varepsilon$  the counit of  $\mathcal{U}$ .

In Section 4, we show how to reconstruct a realization of the process on a Bose-Fock space using quantum stochastic calculus. With the help of the explicit expressions for the Cartan elements given in Proposition 4.2, we can give a classical stochastic process in Section 5, whose joint moments coincide with the vacuum expectation of the restriction of the process to the Cartan subalgebra (see Theorem 5.1). This process is seen to be a Poisson jump process on the lattice generated by the weights of the unitary irreducible representations of  $\mathcal{U}$ . We also give several other elements of  $\mathcal{U}$ , whose vacuum expectations can be characterised by classical processes.

## 2 Preliminaries

### 2.1 The Hopf algebras $U_q(\mathfrak{g})$

Let  $\mathfrak{g}$  be any complex simple Lie algebra and  $(a_{ij})_{1 \leq i, j \leq n}$  be its Cartan matrix. Let  $(d_i)_{1 \leq i \leq n}$  be non-zero integers such that  $d_i a_{ij} = d_j a_{ji}$  and the greatest common divisor of the  $d_i$ 's is 1. Let furthermore  $q \neq 0$  be a complex number such that  $q^{2d_i} \neq 1$  for all  $i$ . The quantum enveloping algebra  $U_q(\mathfrak{g})$  is defined [J85] as the Hopf algebra generated by  $e_i, f_i, k_i$ , and  $k_i^{-1}$ ,  $i = 1, \dots, n$  with the relations

$$\begin{aligned} k_i k_i^{-1} &= k_i^{-1} k_i = 1, & k_i k_j &= k_j k_i, \\ k_i e_j &= q_i^{a_{ij}/2} e_j k_i, & k_i f_j &= q_i^{-a_{ij}/2} f_j k_i, \\ e_i f_j - f_j e_i &= \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q_i - q_i^{-1}}, \\ \sum_{n=0}^{1-a_{ij}} \binom{1-a_{ij}}{n}_{q_i} (-1)^n e_i^{1-a_{ij}-n} e_j e_i^n &= 0 & \text{for } i \neq j \\ \sum_{n=0}^{1-a_{ij}} \binom{1-a_{ij}}{n}_{q_i} (-1)^n f_i^{1-a_{ij}-n} f_j f_i^n &= 0 & \text{for } i \neq j \\ \Delta(e_i) &= e_i \otimes k_i^{-1} + k_i \otimes e_i, & \Delta(f_i) &= f_i \otimes k_i^{-1} + k_i \otimes f_i, & \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\ \varepsilon(e_i) &= \varepsilon(f_i) = 0, & \varepsilon(k_i^{\pm 1}) &= 1, \\ S(e_i) &= -q_i^{-1} e_i, & S(f_i) &= -q_i f_i, & S(k_i) &= k_i^{-1}, \end{aligned}$$

where  $q_i = q^{d_i}$  and  $\binom{n}{\nu}_q$  is defined by

$$\binom{n}{\nu}_q = \frac{[n]_q!}{[\nu]_q! [n-\nu]_q!}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We restrict ourselves to  $q \in \mathbb{R} \setminus \{0\}$  and also define an anti-involution on  $U_q(\mathfrak{g})$  which is given on the generators by

$$(e_i)^* = f_i, \quad (f_i)^* = e_i, \quad (k_i)^* = k_i,$$

The implementation of this anti-involution produces the quantum enveloping algebras  $U_q(\mathfrak{g}_c)$  corresponding to the simple compact Lie algebras  $\mathfrak{g}_c$ . We shall use the notation:  $\mathcal{U} \equiv U_q(\mathfrak{g}_c)$ .

### 2.2 Lévy processes on bialgebras

We recall the definition of Lévy processes on  $*$ -bialgebras, cf. [Sch93].

**Definition 2.1** A family of  $*$ -homomorphisms  $(j_{st})_{0 \leq s \leq t}$  defined on a  $*$ -bialgebra  $\mathcal{B}$  with values in another  $*$ -algebra  $\mathcal{A}$  with some fixed state  $\Phi : \mathcal{A} \rightarrow \mathbb{C}$  is called a Lévy process (w.r.t.  $\Phi$ ), if the following conditions are satisfied:

- (i) the images corresponding to disjoint time intervals commute, i.e.  $[j_{st}(\mathcal{B}), j_{s't'}(\mathcal{B})] = \{0\}$  for  $0 \leq s \leq t \leq s' \leq t'$ , and expectations corresponding to disjoint time intervals factorize, i.e.

$$\Phi(j_{s_1 t_1}(b_1) \cdots j_{s_n t_n}(b_n)) = \Phi(j_{s_1 t_1}(b_1)) \cdots \Phi(j_{s_n t_n}(b_n)),$$

for all  $n \in \mathbb{N}$ ,  $b_1, \dots, b_n \in \mathcal{B}$  and  $0 \leq s_1 \leq t_1 \leq \dots \leq t_n$ ;

- (ii)  $m_{\mathcal{A}} \circ (j_{st} \otimes j_{tu}) \circ \Delta = j_{su}$  for all  $0 \leq s \leq t \leq u$ ;
- (iii) the functionals  $\varphi_{st} = \Phi \circ j_{st} : \mathcal{B} \rightarrow \mathbb{C}$  depend only on  $t - s$ ;
- (iv)  $\lim_{t \searrow s} j_{st}(b) = j_{ss}(b) = \varepsilon(b)1_{\mathcal{A}}$  for all  $b \in \mathcal{B}$ .

For a detailed exposition of the theory of these processes see [Sch93], for a more accessible first introduction see also [M95, Chapter VII] or [Sch91].

The functionals  $\varphi_{t-s} = \varphi_{st}$  then form a convolution semi-group of states and there exists a hermitian conditionally positive (i.e. positive on  $\ker \varepsilon$ ) linear functional  $\psi$  such that  $\varphi_t = \exp_{\star} t\psi = \varepsilon + t\psi + \frac{t^2}{2}\psi \star \psi + \dots + \frac{t^n}{n!}\psi^{\star n} + \dots$  and  $\psi(1) = 0$ . Conversely, for every hermitian conditionally positive linear functional  $\psi : \mathcal{B} \rightarrow \mathbb{C}$  with  $\psi(1) = 0$  there exists a unique convolution semi-group of states  $(\varphi_t)_{t \in \mathbb{R}_+}$  and a unique (up to equivalence) Lévy process  $(j_{st})_{0 \leq s \leq t}$ . The functional  $\psi$  is called the generator of the Lévy process  $(j_{st})_{0 \leq s \leq t}$ .

**Definition 2.2** Two quantum stochastic processes (i.e. families of  $*$ -homomorphisms)  $(j_{st}^{(1)} : \mathcal{B} \rightarrow \mathcal{A}_1)$  and  $(j_{st}^{(2)} : \mathcal{B} \rightarrow \mathcal{A}_2)$  on the same  $*$ -bialgebra  $\mathcal{B}$  are called equivalent (with respect to two fixed states  $\Phi_1$  and  $\Phi_2$  on  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively) if their joint moments agree, i.e. if

$$\Phi_1(j_{s_1 t_1}^{(1)}(b_1) \cdots j_{s_n t_n}^{(1)}(b_n)) = \Phi_2(j_{s_1 t_1}^{(2)}(b_1) \cdots j_{s_n t_n}^{(2)}(b_n)),$$

for all  $n \in \mathbb{N}$ ,  $b_1, \dots, b_n \in \mathcal{B}$  and  $s_1, \dots, s_n, t_1, \dots, t_n \in \mathbb{R}_+$ .

Let us now show that Lévy processes give indeed infinitely divisible representations, as we claimed in the Introduction. Let  $\pi_{st} = (\pi_{st}, H_{st}, \Omega_{st})$  be the GNS representation induced by  $j_{st} : \mathcal{B} \rightarrow \mathcal{A}$ , i.e.  $H_{st}$  is the pre-Hilbert space obtained by taking the quotient of  $\mathcal{B}$  with respect to the null space  $\mathcal{N}_{st} = \{a \in \mathcal{B}; \Phi(j_{st}(a^*a)) = 0\}$  with the inner product induced from the inner product  $\langle a, b \rangle = \Phi(j_{st}(a^*b))$  on  $\mathcal{B}$ ,  $\pi_{st}$  is the representation of  $\mathcal{B}$  on  $H_{st} = \mathcal{B}/\mathcal{N}_{st}$  induced from left multiplication, and  $\Omega_{st}$  is the image of the unit element 1 under the canonical projection from  $\mathcal{B}$  to  $H_{st}$ . Then, by property (iii) of Definition 2.1,

$(\pi_{st} : \mathcal{B} \rightarrow L(H_{st}))$  is equivalent to  $(j_{s't'} : \mathcal{B} \rightarrow \mathcal{A})$  (with respect to the states  $\langle \Omega_{st}, \pi_{st}(\cdot) \Omega_{st} \rangle$  and  $\Phi$  on  $L(H_{st})$  and  $\mathcal{A}$ , respectively) if the intervals  $(s, t)$  and  $(s', t')$  have the same length. The product of such representations  $\pi_{s_1 t_1}, \dots, \pi_{s_n t_n}$  is therefore by property (ii) equivalent to  $j_{s_1 + \dots + s_n, t_1 + \dots + t_n} \cong \pi_{s_1 + \dots + s_n, t_1 + \dots + t_n}$ . Therefore, if we want to write  $j_{st} \cong \pi_{st}$  as an  $n$ -fold product

$$\pi_{st} \cong \prod_{i=1}^n \pi_i, \quad \text{with} \quad \pi_i = \pi^{(n)} \quad \text{for } i = 1, \dots, n,$$

it is sufficient to take  $\pi^{(n)} = \pi_{(t-s)/n}$ . This proves that  $\pi_{st}$  is indeed infinitely divisible.

For a given generator  $\psi$  one can construct the so-called Schürmann triple  $(\rho, \eta, \psi)$  consisting of a unitary representation  $\rho$  of  $\mathcal{B}$  on some pre-Hilbert space  $D$ , a  $(\rho, \eta)$ -1-cocycle  $\eta : \mathcal{B} \rightarrow D$ , and the generator  $\psi : \mathcal{B} \rightarrow C$  itself, such that the following relations

$$\begin{aligned} \eta(ab) &= \rho(a)\eta(b) + \eta(a)\varepsilon(b), \\ \langle a, b \rangle &= -\varepsilon(a^*)\psi(b) + \psi(a^*b) - \psi(a^*)\varepsilon(b), \end{aligned}$$

hold for all  $a, b \in \mathcal{B}$ .

This construction goes as follows. First we define a sesqui-linear form on  $\mathcal{B}$  by

$$\langle a, b \rangle_\psi = \psi((a - \varepsilon(a)1)^*(b - \varepsilon(b)1)),$$

for  $a, b \in \mathcal{B}$ . Since  $a \mapsto a - \varepsilon(a)1$  is a projection from  $\mathcal{B}$  to  $\ker \varepsilon$  and since  $\psi$  is positive on  $\ker \varepsilon$ , this form is positive semi-definite. If we quotient  $\mathcal{B}$  by the nullspace of this form,

$$\mathcal{N}_\psi = \{b \in \mathcal{B}; \langle a, a \rangle_\psi = 0\},$$

then we obtain a pre-Hilbert space  $D = \mathcal{B}/\mathcal{N}_\psi$ , this will be the space on which the representation  $\rho$  acts. The cocycle  $\eta$  is just the canonical projection from  $\mathcal{B}$  onto  $D$ , and the inner product on  $D$  and the sesqui-linear form on  $\mathcal{B}$  are related via

$$\langle \eta(a), \eta(b) \rangle = \langle a, b \rangle_\psi = -\varepsilon(a^*)\psi(b) + \psi(a^*b) - \psi(a^*)\varepsilon(b),$$

for  $a, b \in \mathcal{B}$ . Since  $\mathcal{N}_\psi \cap \ker \varepsilon$  is invariant under left multiplication of elements of  $\ker \varepsilon$ , we have an action of  $\ker \varepsilon$  on  $\ker \varepsilon / (\mathcal{N}_\psi \cap \ker \varepsilon) = D$  such that  $\rho(a)\eta(b) = \eta(ab)$  for all  $a, b \in \ker \varepsilon$ . This representation can be extended to a representation of  $\mathcal{B}$  on  $D$ , if we set

$$\rho(a)\eta(b) = \eta(ab) - \eta(a)\varepsilon(b),$$

for  $a, b \in \mathcal{B}$ . If we require the cocycle  $\eta$  to be onto, then the Schürmann triple is unique up to isometry.

We will do this construction in the reversed sense in Section 3 in order to classify the generators of Lévy processes on the  $*$ -Hopf algebras  $\mathcal{U}$  starting from the classification of their unitary representations.

### 3 Classification of the generators

We will now give a complete classification of the generators of Lévy processes on  $\mathcal{U}$ . We begin by proving a series of lemmas which we shall use to formulate our main result.

**Lemma 3.1** *Define the following ideals:  $K_1 = \ker \varepsilon$  and  $K_2 = K_1 \cdot K_1 = \text{span}\{uv; u, v \in K_1\}$ . Then  $K_2 = K_1$ .*

**Proof:**  $K_1$  is generated by  $e_i, f_i, k_i - 1$  and  $k_i^{-1} - 1$ . The relation  $k_i e_j = q_i^{a_{ij}/2} e_j k_i$  implies  $e_j \in K_2$  since it can be written as

$$e_j = \frac{(k_i - 1)e_j(k_i^{-1} - 1) + (k_i - 1)e_j + e_j(k_i^{-1} - 1)}{q_i^{a_{ij}/2} - 1}$$

if we choose  $i$  such that  $a_{ij} \neq 0$ . Similarly, we get  $f_j \in K_2$ . And  $k_1 - 1, k_1^{-1} - 1 \in K_2$  follows from

$$(q_i - q_i^{-1})[e_i, f_i] - (k_i - 1)^2 + (k_i^{-1} - 1)^2 = 2(k_i - k_i^{-1}) \in K_2,$$

since  $k_i - 1 = \frac{1}{2} (k_i(k_i - k_i^{-1}) - (k_i - 1)^2)$  and  $k_i^{-1} - 1 = -\frac{1}{2} (k_i^{-1}(k_i - k_i^{-1}) + (k_i^{-1} - 1)^2)$ . ■

**Lemma 3.2** *Suppose we have a second order Casimir element  $C$  in  $\mathcal{U}$  such that for all unitary irreducible representations  $\pi$  of  $\mathcal{U}$  except the one-dimensional  $\pi(C) - \varepsilon(C)$  is invertible. Let  $\rho$  be an arbitrary unitary representation of  $\mathcal{U}$  on some pre-Hilbert space  $D$ . Then all  $(\rho, \varepsilon)$ -1-cocycles are trivial, i.e. there exists a vector  $\Omega_\eta \in D$  such that*

$$\eta(u) = (\rho(u) - \varepsilon(u))\Omega_\eta \quad \text{for all } u \in \mathcal{U}.$$

**Proof:** The cocycle equation  $\eta(uv) = \rho(u)\eta(v) + \eta(u)\varepsilon(v)$  implies  $\eta(1) = 0$ , so that it is sufficient to determine  $\eta$  on  $K_1 = \ker \varepsilon$ .

The representation  $\rho$  is direct sum of the unitary irreducible ones,  $\rho = \sum_{\lambda \in \Lambda_\rho} \pi_\lambda$ ,  $D = \sum_{\lambda \in \Lambda_\rho} D_\lambda$ . Let  $\Omega_\eta$  be the unique vector in  $D$  that satisfies

$$\eta(C) = (\rho(C) - \varepsilon(C))\Omega_\eta,$$

and that has no component in the one-dimensional representations. We now have to show that  $\eta$  is equal to the cocycle defined by  $\Omega_\eta$ .

$uC = Cu$  for all  $u \in \mathcal{U}$  implies

$$\rho(u)\eta(C) + \eta(u)\varepsilon(C) = \eta(uC) = \eta(Cu) = \rho(C)\eta(u) + \eta(C)\varepsilon(u)$$

and thus  $(\rho(C) - \varepsilon(C))\eta(u) = \rho(u)\eta(C)$  for  $u \in K_1$ . Therefore  $\eta(u) - \rho(u)\Omega_\eta$  has to be contained in the one-dimensional components of  $D = \sum_{\lambda \in \Lambda_\rho} D_\lambda$  and the cocycle  $\bar{\eta}$  defined by  $\bar{\eta}(u) = \eta(u) - \rho(u)\Omega_\eta$  for  $u \in \mathcal{U}$  is an  $(\varepsilon, \varepsilon)$ -1-cocycle. This implies that  $\bar{\eta}$  is zero on  $K_2$ , and, by Lemma 3.1, on  $K_1$ , and thus on all of  $\mathcal{U}$ . ■

**Lemma 3.3** *The generator  $\psi : \mathcal{U} \rightarrow \mathbb{C}$  is uniquely determined by  $\rho$  and  $\eta$ .*

**Proof:** This follows immediately from Lemma 3.1, since  $\psi(1) = 0$  and  $\mathcal{U} = \mathbb{C}1 \oplus K_1$ . ■

**Lemma 3.4** *There exists a second order Casimir element  $C$  in  $\mathcal{U}$  such that for all unitary irreducible representations  $\pi$  of  $\mathcal{U}$  except the one-dimensional  $\pi(C) - \varepsilon(C)$  is invertible.*

**Proof:** First we recall that there exists a second order Casimir element  $C_2$  in  $U_q(\mathfrak{g})$ , cf. e.g., [S83, J85, J86]. For all finite-dimensional highest weight representations of  $U_q(\mathfrak{g})$ , except for the one-dimensional  $\pi_{\text{id}}$ , we have  $\pi(C_2) \neq \varepsilon(C_2)$ . This is so since  $\pi(C_2) \neq \pi_{\text{id}}(C_2) (= \varepsilon(C_2))$  if  $\pi \neq \pi_{\text{id}}$ . The last fact follows from the results of Rosso [R88], but can also be obtained by supposing the inverse and then getting a contradiction in the limit  $q \rightarrow 1$ . The same facts hold for  $C_2^*$  - the image of  $C_2$  under the involution producing  $\mathcal{U}$ , and also for  $C = (C_2 + C_2^*)/2$ . Now it only remains to note that  $C$  is invariant under the involution and thus is the required Casimir of  $\mathcal{U}$ , and that the unitary irreducible representations of  $\mathcal{U}$  are in 1-to-1 correspondence with the finite-dimensional highest weight representations of  $U_q(\mathfrak{g})$  and are obtained from the latter by use of the involution. ■

We summarize these results in the following theorem.

**Theorem 3.5** *Every generator on  $\mathcal{U}$  is of the form*

$$\psi(u) = \langle \Omega, (\rho(u) - \varepsilon(u))\Omega \rangle, \quad \text{for all } u \in \mathcal{U},$$

where  $\rho$  is a unitary representation of  $\mathcal{U}$  on some pre-Hilbert space  $D$  and  $\Omega$  some vector in  $D$ .

**Remark:** The correspondence between generators  $\psi$  and Schürmann triples  $(\rho, \eta, \psi)$  is 1-to-1, if we require  $\eta$  to be surjective. Furthermore, for a given cocycle  $\eta$  the choice of  $\Omega_\eta$  is unique, if we demand that  $\Omega_\eta$  has no components in the trivial one-dimensional representations. Therefore the correspondence between Lévy processes on  $\mathcal{U}$  and triples  $(\rho, D, \Omega)$  consisting of a unitary representation  $\rho$  on a pre-Hilbert space  $D$  and a vector  $\Omega \in D$  becomes a bijection, if we impose the following two conditions:

1. The trivial one-dimensional representation does not appear in the direct sum decomposition  $\rho = \sum_{\lambda \in \Lambda_\rho} \pi_\lambda$ ,  $D = \sum_{\lambda \in \Lambda_\rho} D_\lambda$  of  $(\rho, D)$ .
2. The vector  $\Omega \in D$  is cyclic for  $(\rho, D)$ , i.e.  $D = \rho(\mathcal{U})\Omega$ .

## 4 Construction of the processes

From Theorem 3.5 we know that every generator on  $\mathcal{U}$  is given by a unitary representation  $\rho$  on a pre-Hilbert space  $D$  and a vector  $\Omega \in D$ . From the general theory follows that the corresponding process can be constructed on the Fock space  $\Gamma(L^2(\mathbb{R}_+, H))$  (or rather on a dense stable subspace thereof), where  $H$  is the Hilbert space closure of  $D$ , via a quantum stochastic differential equation.

More precisely, we have the following theorem.

**Theorem 4.1** *Let  $\rho$  be a unitary representation of  $\mathcal{U}$  on  $D$  and let  $\Omega \in D$ . Set  $\eta(u) = (\rho(u) - \varepsilon(u))\Omega$ ,  $\tilde{\eta}(u) = \eta(u^*)$ ,  $\psi(u) = \langle \Omega, (\rho(u) - \varepsilon(u))\Omega \rangle$  for  $u \in \mathcal{U}$ . Then the quantum stochastic differential equations*

$$dj_{st}(u) = m \circ \left( j_{st} \otimes (d\Lambda(\rho - \varepsilon) + dA^*(\eta) + dA(\tilde{\eta}) + \psi dt) \right) (\Delta u),$$

*with the initial conditions  $j_{ss}(u) = \varepsilon(u)\text{id}$  for  $u \in \mathcal{U}$  have solutions on a domain  $\mathcal{E}_D \subseteq \Gamma(L^2(\mathbb{R}_+, H))$  that contains the Fock vacuum, that is dense in the Fock space  $\Gamma(L^2(\mathbb{R}_+, H))$  and that is invariant under  $j_{st}(\mathcal{U})$  for all  $s \leq t$ ,  $s, t \in \mathbb{R}_+$ .*

*Furthermore, in the vacuum state  $(j_{st})$  is a Lévy process on  $\mathcal{U}$  with generator  $\psi$ .*

**Proof:** The triple  $(\rho, \eta, \psi)$  satisfies the conditions of [Sch93, Theorem 2.3.5], therefore our theorem follows from Schürmann's representation theorem (see [Sch93, Theorem 2.3.5] and [Sch93, Theorem 2.5.3]). For the exact definition of the domain  $\mathcal{E}_D$ , see Page 44 of [Sch93].  $\blacksquare$

For group-like elements (e.g. the  $k_i$ 's) we can use [Sch93, Proposition 4.1.2] to get explicit expressions without having to solve any quantum stochastic differential equations. These expressions become particularly simple, if we act on the exponential or coherent vectors

$$\mathcal{E}(f) = \sum_{n \in \mathbb{N}} \frac{f^{\otimes n}}{\sqrt{n!}}$$

for  $f \in L^2(\mathbb{R}_+) \otimes D$ .

**Proposition 4.2** *Let  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ ,  $k^\lambda = k_1^{\lambda_1} \dots k_n^{\lambda_n}$ ,  $f \in L^2(\mathbb{R}_+) \otimes D$ , and  $(j_{st})$  be the process defined in Theorem 4.1 for the triple  $(\rho, \eta, \psi)$ . Then we have*

$$\begin{aligned} & j_{st}(k^\lambda) \mathcal{E}(f) \\ &= \exp \left( (t-s)\psi(k^\lambda) + \int_s^t \langle \eta(k^\lambda), f(r) \rangle dr \right) \mathcal{E} \left( f \mathbb{1}_{[0, s[ \cup [t, \infty[} + (\rho(k^\lambda)f + \eta(k^\lambda)) \mathbb{1}_{[s, t]} \right). \end{aligned}$$

*In particular, in the Fock vacuum we get*

$$\varphi_{t-s}(k^\lambda) = \langle \mathcal{E}(0), j_{st}(k^\lambda) \mathcal{E}(0) \rangle = \exp \left( (t-s)\psi(k^\lambda) \right).$$



## 5 Classical processes

In this section we will assume  $q > 0$ .

The Cartan elements  $k_i, k_i^{-1}$ ,  $i = 1, \dots, n$ , generate a commutative sub-Hopf  $*$ -algebra  $\mathcal{K}$  of  $\mathcal{U}$ . Therefore the restriction of  $(j_{st})$  to  $\mathcal{K}$  is still a Lévy process. Furthermore, due to the self-adjointness of the  $k_i$  and the commutativity of  $\mathcal{K}$ , there exists a classical version of  $(j_{st}(k_1), \dots, j_{st}(k_n))$ , i.e. a real-valued stochastic process  $(\hat{k}_1(s, t), \dots, \hat{k}_n(s, t))$  such that

$$\Phi(j_{s_1 t_1}(k^{\mu_1}) \cdots j_{s_m t_m}(k^{\mu_m})) = \mathbb{E}(\hat{k}^{\mu_1}(s_1, t_1) \cdots \hat{k}^{\mu_m}(s_m, t_m)) \quad (1)$$

holds for all  $m \in \mathbb{N}$ ,  $\mu_1, \dots, \mu_m \in \mathbb{Z}^n$ ,  $s_1, \dots, s_m, t_1, \dots, t_m \in \mathbb{R}_+$ . Using Proposition 4.2 we can explicitly characterize the process  $(\hat{k}_1(s, t), \dots, \hat{k}_n(s, t))$ .

We know that  $\psi$  is of the form  $\psi(u) = \langle \Omega, (\rho(u) - \varepsilon(u))\Omega \rangle$  for all  $u \in \mathcal{U}$  with some unitary representation  $\rho$  acting on a pre-Hilbert space  $D$  and some vector  $\Omega \in D$ .  $D$  is a direct sum of the unitary irreducible representation of  $\mathcal{U}$  and can be decomposed into a direct sum of eigenspaces  $D = \bigoplus_{\kappa} E_{\kappa}$  of the Cartan elements  $k_1, \dots, k_n$ . Furthermore, we know that the eigenvalues are of the form  $\kappa = (q^{\lambda_1/2}, \dots, q^{\lambda_n/2})$  with  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ . Develop  $\Omega$  into a sum of eigenvectors

$$\Omega = \sum_{\lambda \in \Lambda_{\Omega}} v_{\lambda}, \quad \text{with} \quad \rho(k_i)v_{\lambda} = q^{\lambda_i/2}v_{\lambda}$$

and set  $c_{\lambda} = \|v_{\lambda}\|^2$ . Then we get

$$\psi(k^{\mu}) = \sum_{\lambda \in \Lambda_{\Omega}} c_{\lambda} (q^{\langle \mu, \lambda \rangle / 2} - 1)$$

for all  $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ . From Proposition 4.2 we can now deduce the moments of  $(\hat{k}_1(s, t), \dots, \hat{k}_n(s, t))$ , we get

$$\Phi(j_{st}(k^{\mu})) = \exp \left( (t - s) \sum_{\lambda \in \Lambda_{\Omega}} c_{\lambda} (q^{\langle \mu, \lambda \rangle / 2} - 1) \right)$$

for  $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ . We see that we can give  $(\hat{k}_1(s, t), \dots, \hat{k}_n(s, t))$  as a function of a Poisson process on the lattice generated by the elements of  $\Lambda_{\Omega}$ .

**Theorem 5.1** *Let  $\{(N_t^{(\lambda)})_{t \in \mathbb{R}_+}; \lambda \in \Lambda_{\Omega}\}$ , be a family of independent Poisson processes, and define a jump process  $(N_t)_{t \in \mathbb{R}_+} = ((N_1(t), \dots, N_n(t)))_{t \in \mathbb{R}_+}$  with values in the lattice generated by the set  $\Lambda_{\Omega}$  by*

$$N_t = \sum_{\lambda \in \Lambda_{\Omega}} \lambda N_{c_{\lambda} t}, \quad t \in \mathbb{R}_+.$$

Then

$$(\hat{k}_1(s, t), \dots, \hat{k}_n(s, t)) = (q^{(N_1(t) - N_1(s))/2}, \dots, q^{(N_n(t) - N_n(s))/2})$$

is a classical version of  $(j_{st}(k_1), \dots, j_{st}(k_n))$ .

**Remark:** We can also give the following equivalent construction of  $(N_t)$ . Set  $c = \sum_{\lambda \in \Lambda_\Omega} c_\lambda$ , i.e.  $c = \|\Omega\|^2$ . Let  $(T_i)_{i \in \mathbb{N}}$  be a sequence of independent, identically distributed (i.i.d.) random variables with exponential distribution with parameter  $c$ , i.e.  $\mathbb{P}(T_i < \tau) = 1 - e^{-c\tau}$  for  $\tau \geq 0$ , and let  $(\Delta_i)_{i \in \mathbb{N}}$  be independent, identically distributed random variables, independent of  $(T_i)$ , with values in  $\Lambda_\Omega$ , such that  $\mathbb{P}(\Delta_i = \lambda) = c_\lambda/c$ . Then we can define  $(N_t)$  as

$$N_t = \sum_{i=1}^{\infty} \Delta_i \mathbb{I}_{\{\sum_{l=1}^i T_l \leq t\}}, \quad t \in \mathbb{R}_+.$$

**Proof:** We have to show that Equation (1) is satisfied for all  $m \in \mathbb{N}$ ,  $\mu_1, \dots, \mu_m \in \mathbb{Z}^n$ ,  $s_1, \dots, s_m, t_1, \dots, t_m \in \mathbb{R}_+$ . Without loss of generality we can assume that  $s_1 \leq t_1 \leq s_2 \leq \dots \leq t_m$ . Using the independence of the increments, we get

$$\begin{aligned} \Phi(j_{s_1 t_1}(k^{\mu_1}) \dots j_{s_m t_m}(k^{\mu_m})) &= \Phi(j_{s_1 t_1}(k^{\mu_1})) \dots \Phi(j_{s_m t_m}(k^{\mu_m})) \\ &= \prod_{l=1}^m \exp \left( (t_l - s_l) \sum_{\lambda \in \Lambda_\Omega} c_\lambda (q^{\langle \mu_l, \lambda \rangle / 2} - 1) \right) \\ &= \prod_{l=1}^m \mathbb{E} \left( q^{\frac{1}{2} \sum_{\lambda \in \Lambda_\Omega} \langle \mu_l, \lambda \rangle (N_{c_\lambda t_l}^{(\lambda)} - N_{c_\lambda s_l}^{(\lambda)})} \right) \\ &= \mathbb{E} \left( \hat{k}^{\lambda_1}(s_1, t_1) \dots \hat{k}^{\lambda_m}(s_m, t_m) \right). \end{aligned}$$

■

Comparing the right-hand-side and the left-hand-side of Equation (1), we get the following result.

**Corollary 5.2** *Let  $\mathcal{E}(0) = \sum_{\lambda \in \Lambda_{j_{st}}} v_\lambda(s, t)$  be a decomposition of the Fock vacuum into a sum of joint eigenvectors of  $j_{st}(k_1), \dots, j_{st}(k_n)$ , i.e.  $j_{st}(k_i)v_\lambda(s, t) = q^{\lambda_i/2}v_\lambda(s, t)$ . Then the norms of the  $v_\lambda(s, t)$  are given by*

$$\|v_\lambda(s, t)\|^2 = \mathbb{P}(N_t = \lambda), \quad \lambda \in \Lambda_{j_{st}}.$$

The Casimir elements also give commuting families of operators and therefore classical processes.

**Proposition 5.3** *Let  $C$  be the self-adjoint second order Casimir element. (cf. Lemma 3.4). Then  $(j_{0t}(C))$  has a classical version, i.e. there exists a real-valued stochastic process  $(\hat{C}(t))_{t \in \mathbb{R}_+}$  such that all joint moments of  $(j_{0t}(C))$  agree with those of  $(\hat{C}(t))$ .*

More generally, let  $C_1, \dots, C_r$  be Casimir operators, i.e. elements of the center  $\mathcal{Z}(\mathcal{U})$  of  $\mathcal{U}$ , such that  $C_i^* = C_i$  for  $i = 1, \dots, r$ . Then there exists a classical version of  $(j_{0t}(C_1), \dots, j_{0t}(C_r))_{t \in \mathbb{R}_+}$ , i.e. an  $\mathbb{R}^r$ -valued stochastic process with the same joint moments.

**Proof:** This follows from [Sch93, Proposition 4.2.3] (see also [F99, Theorem 2.3] for the multi-dimensional version). The commutation relations  $[C_i \otimes 1, \Delta(C_j)] = 0$  for  $i, j = 1, \dots, r$  imply that  $[j_{0s}(C_i), j_{0t}(C_j)] = j_{0s} \otimes j_{st}([C_i \otimes 1, \Delta(C_j)]) = 0$  for  $0 \leq s \leq t$ , and therefore that the operators of the family  $(j_{0t}(C_1), \dots, j_{0t}(C_r))_{t \in \mathbb{R}_+}$  commute. Since they are symmetric, their joint moments are positive and there exists a (not necessarily unique) solution of the associated moment problem. Any classical process whose distribution is given by such a solution of the moment problem is a classical version of  $(j_{0t}(C_1), \dots, j_{0t}(C_r))_{t \in \mathbb{R}_+}$ . ■

For the Lie algebra  $u(n)$  and the Casimir operators

$$G_m^n = \sum_{1 \leq i_1 < \dots < i_m \leq n} \sum_{\pi, \sigma \in \mathcal{S}_m} \text{sgn}(\pi\sigma) E_{i_{\pi(1)} i_{\sigma(1)}} \cdots E_{i_{\pi(m)} i_{\sigma(m)}}$$

of “determinant type”, the process  $(\mathcal{G}_m^n(t) = j_{0t}(G_m^n))_{1 \leq m \leq n, t \geq 0}$  (for a certain Lévy process  $(j_{st})$  on  $U(u(n))$ ) has been considered by Hudson and Parthasarathy in [HP94]. But a characterization of the classical process associated to this operator process is not known even for this special case, as far as we know. It would be interesting to know more about these processes. For a special process on  $U_q(su(2))$  with a relatively simple generator the generator of the classical version was determined in [F99].

We can find another element that gives us a classical process.

**Proposition 5.4** *Let  $i \in \{1, \dots, n\}$  and set  $Z = k_i^{-1}e_i + f_i k_i^{-1}$ . Then  $(j_{0t}(z))$  has a classical version, i.e. there exists a real-valued stochastic process  $(\hat{Z}_t)$  such that*

$$\Phi(j_{0t_1}(Z^{\mu_1}) \cdots j_{0t_m}(Z^{\mu_m})) = \mathbb{E}(\hat{Z}_{t_1}^{\mu_1} \cdots \hat{Z}_{t_m}^{\mu_m})$$

for all  $m \in \mathbb{N}$ ,  $\mu_1, \dots, \mu_m \in \mathbb{N}$ ,  $t_1, \dots, t_m \in \mathbb{R}_+$ .

**Proof:** As in the preceding proposition the existence of the classical version follows from Schürmann’s criterium [Sch93, Proposition 4.2.3], since  $Z \otimes 1$  and  $\Delta Z = Z \otimes k_i^{-2} + 1 \otimes Z$  commute. ■

## 6 Outlook

In this paper we have discussed Lévy processes on the compact forms  $\mathcal{U}$ . Further, we plan to investigate what is the most general class of Hopf algebras, on which the arguments work and analogous results can be obtained. Naturally, we shall look first at the quantum enveloping algebras  $U_q(\mathfrak{g}_n)$  corresponding to the non-compact semisimple Lie algebras  $\mathfrak{g}_n$ . Already this setting is rather involved. We shall point out only some complications. One issue is related to the phenomena, that unlike the compact forms  $U_q(\mathfrak{g}_c)$  which are 1-to-1 with  $\mathfrak{g}_c$ , there are several

possible quantum enveloping algebras corresponding to a non-compact semisimple Lie algebra  $\mathfrak{g}_n$ . Moreover, the different procedures to construct  $U(\mathfrak{g}_n)$  lead to different results, cf. e.g., [L91, D91, T92]. After one has selected the deformation of the real form one would encounter the next difficulty related to the fact that the Casimir operators are not separating the irreducible representations well enough. Namely, there are inequivalent irreducible representations which share the same values of the Casimir operators. These are irreducible representations which are subrepresentations of reducible partially equivalent generalized principle series representations. In the classical case this phenomenon is explained in, e.g., [D77, D88], and references therein.

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